An Example of Using Taylor Table To Derive a 4^{th} Order Compact Pade Scheme

The generalized form of the equation is given by

$$d\left(\frac{\partial u}{\partial x}\right)_{j-1} + \left(\frac{\partial u}{\partial x}\right)_{j} + e\left(\frac{\partial u}{\partial x}\right)_{j+1} - \frac{1}{\Delta x}(au_{j-1} + bu_j + cu_{j+1}) = er_t$$

The equation is written on terms of coefficients a, b, c, d, e (the coefficient on the j point is taken as one to simplify the algebra) which must be determined using the Taylor table approach as outlined below.

The Taylor table is

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5 \cdot \left(\frac{\partial^5 u}{\partial x^5}\right)_j$
$\Delta x \cdot d\left(\frac{\partial u}{\partial x}\right)_{j-1}$		d	$d \cdot (-1) \cdot \frac{1}{1!}$	$d \cdot (-1)^2 \cdot \frac{1}{2!}$	$d \cdot (-1)^3 \cdot \frac{1}{3!}$	$d \cdot (-1)^4 \cdot \frac{1}{4!}$
$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_{j}$		1				
$\Delta x \cdot e\left(\frac{\partial u}{\partial x}\right)_{j+1}$		e	$e \cdot (1) \cdot \frac{1}{1!}$	$e \cdot (1)^2 \cdot \frac{1}{2!}$	$e \cdot (1)^3 \cdot \frac{1}{3!}$	$e \cdot (1)^4 \cdot \frac{1}{4!}$
$-a \cdot u_{j-1}$	$-a$	$-a \cdot (-1) \cdot \frac{1}{1!}$	$-a \cdot (-1)^2 \cdot \frac{1}{2!}$	$-a \cdot (-1)^3 \cdot \frac{1}{3!}$	$-a \cdot (-1)^4 \cdot \frac{1}{4!}$	$-a \cdot (-1)^5 \cdot \frac{1}{5!}$
$-b \cdot u_j$	$-b$					
$-c \cdot u_{j+1}$	$-c$	$-c \cdot (1) \cdot \frac{1}{1!}$	$-c \cdot (1)^2 \cdot \frac{1}{2!}$	$-c \cdot (1)^3 \cdot \frac{1}{3!}$	$-c \cdot (1)^4 \cdot \frac{1}{4!}$	$-c \cdot (1)^5 \cdot \frac{1}{5!}$
$=$						
$\Delta x er_t$	0	0	0	0	0	?

To maximize the order of accuracy, we must set the the first five columns to zero producing the matix equation for the coefficients,

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -2 & 2 \\ 1 & 0 & -1 & 3 & 3 \\ -1 & 0 & -1 & -4 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

having the solution $[a, b, c, d, e] = \frac{1}{4}[-3, 0, 3, 1, 1]$. Under these conditions, the sixth column sums to

$$er_t = \frac{\Delta x^4}{120}\left(\frac{\partial^5 u}{\partial x^5}\right)_j$$

and the method can be expressed as

$$\left(\frac{\partial u}{\partial x}\right)_{j-1} + 4\left(\frac{\partial u}{\partial x}\right)_{j} + \left(\frac{\partial u}{\partial x}\right)_{j+1} - \frac{3}{\Delta x}(-u_{j-1} + u_{j+1}) = O(\Delta x^4)$$